Topological versions of Abel-Jacobi, the height pairing, and the Poincaré bundle

Mirel Caibăr   C. Herbert Clemens

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1   Introduction

This note continues the extension of normal function constructions to the topological setting begun in [C]. This time we begin with a topological version of the Abel-Jacobi map for algebraic cycles, a slight variant of the topological version developed in [HLZ] and [DK]. We then go on to construct a topological version of the height pairing for cycles and interpret it as a lifting of (the Abel-Jacobi image of) the pair of cycles to a point in the fiber of the Poincaré bundle. The construction was inspired by the work of Hain [Hai] in the algebraic case. In fact, a one-sentence summary is that we simply unwind Hain's constructions on a principally polarized (intermediate) Jacobian bundle from the underlying complex structure via the Cheeger-Simons theory of differential characters.

In the algebraic setting, the archimedian height pairing of Beilinson, Bloch, Gillet-Soulé is a kind of linking number between two disjoint, homologically trivial cycles of codimension n on a $(2n-1)$-dimensional algebraic manifold. (See, for example, Definition 1.1 of [Wa].) We adapt this notion to topological cycles in a way that makes clearer the behaviour of the height pairing as one of the cycles is moved in such a way as to cross the other. Roughly speaking, given two primitive classes $\eta, \eta' \in H^{2n}(W; \mathbb{Z})$ on a complex manifold W of dimension $2n$ and their associated normal functions

$$\alpha_\eta, \alpha_{\eta'} : \mathbb{L} \to J(X_\mathbb{L}/\mathbb{L})$$

over a Lefschetz pencil $\mathbb{L}$ as in [C], in this note we associate to each $p \in \mathbb{L}$ submanifolds $\Sigma_p, \Sigma'_p \subseteq X_p$ of real dimension $2(n-1)$ whose respective loci, as p transverses $\mathbb{L}$ cut out the Poincaré duals of η and η' respectively. Let $\beta_{\eta'}$ denote the image of $\alpha_{\eta'}$ under the isomorphism

$$J(X_\mathbb{L}/\mathbb{L}) \to J(X_\mathbb{L}/\mathbb{L})^\vee$$

induced by the principal polarization. If $\displaystyle\int_{X_p} \eta \wedge \eta' = 0$ the loci $\{\Sigma_p\}_{p \in \mathbb{L}}$ and $\{\Sigma'_p\}_{p \in \mathbb{L}}$ can be chosen to be disjoint. By the results explained in this note,

this corresponds to the fact that we have a global lifting of the graph of

$$(\alpha_\eta, \beta_{\eta'}) : \mathbb{L} \to J(X_\mathbb{L}) \times_\mathbb{L} J(X_\mathbb{L})^\vee / \mathbb{L}$$

to a never-zero section of the pull-back of the Poincaré bundle, indicating that the Chern class of the pull-back is zero, as it must be by [C]. On the other hand, if $\int_{X_p} \eta \wedge \eta' = m$, then the obstructions to keeping the loci the loci $\{\Sigma_p\}_{p \in \mathbb{L}}$ and $\{\Sigma'_p\}_{p \in \mathbb{L}}$ disjoint can be localized to simple obstructions at m points $p_j \in \mathbb{L}$ such that the X_{p_j} are smooth. Each contributes winding number 1 to the intersection number of η and η'.

We then go on to a more detailed analysis of the case in which Σ and Σ' lie in $CH^n_{hom}(X_p)$. We show that the topological height pairing is simply the real part of a complex invariant whose imaginary part $[\Sigma.\Sigma']$ is the height pairing defined in [GS]. We go on to derive some corollaries of this fact.

The purpose the Part I of this paper is to define a topological height pairing $(\Sigma.\Sigma')$ for homologically trivial cycles on a fixed smooth $X_p = X$ and to show that this height pairing is nothing more nor less than the choice of a section of a topological version of the Poincaré bundle considered as a circle bundle over $J(X) \times J(X)^\vee$. The purpose of Part II of this paper is to treat the case in which Σ and Σ' lie in $CH^n_{hom}(X)$. We show that

$$(\Sigma.\Sigma'(t)) + i[\Sigma.\Sigma'(t)]$$

varies holomorphically for an algebraic family $\{\Sigma'(t)\}_{t \in T}$ and characterize it as a section of the flat line bundle on T given by the incidence divisor of Σ.

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Two notational remarks–all integral homology and cohomology in this paper is taken modulo torsion and, for a cycle Σ on X, $|\Sigma| \subseteq X$ will denote the support of the cycle.

Part I
Topological setting

2 The setting

Let X denote a smooth complex projective manifold of (complex) dimension $2n - 1$ with very ample polarization $\mathcal{O}_X(1)$ and Kähler metric induced by the

associated projective imbedding. Define

$$J(X) = \frac{H_{2n-1}(X;\mathbb{R})}{H_{2n-1}(X;\mathbb{Z})}$$

$$J(X)^\vee = \frac{H^{2n-1}(X;\mathbb{R})}{H^{2n-1}(X;\mathbb{Z})} = Hom\left(H_{2n-1}(X;\mathbb{Z}), \frac{\mathbb{R}}{\mathbb{Z}}\right).$$

Definition 1 *The Poincaré circle bundle is the bundle whose total space is the quotient of the action of $H_{2n-1}(X;\mathbb{Z}) \times H^{2n-1}(X;\mathbb{Z})$ on*

$$H_{2n-1}(X;\mathbb{R}) \times H^{2n-1}(X;\mathbb{R}) \times \frac{\mathbb{R}}{\mathbb{Z}}$$

by the rule that $(m,n) \in H_{2n-1}(X;\mathbb{Z}) \times H^{2n-1}(X;\mathbb{Z})$ acts by

$$(\alpha, \beta, r) \mapsto (\alpha + m, \beta + n, r + n(\alpha) + \beta(m)).$$

Thus

$$d_{J(X)}d_{(J(X)^\vee)}(\beta(\alpha))$$

is the Chern class of the Poincaré bundle.

The justification for the name in the above definition comes from the fact that the restriction of the bundle to

$$J(X) \times \{\beta\}$$

corresponds to a $U(1)$-representation

$$\pi_1(J(X), 0) \to U(1)$$
$$m \mapsto e^{2\pi i \beta(m)}$$

and so to the flat holomorphic line bundle with the above holonomy group.

3 Vanishing cycles, the topological Abel-Jacobi map and the height pairing

In this section we use without citation various standard operator identities in [We], Chapters I and II. Let $\Sigma \subseteq X$ be any smoothly embedded oriented submanifold of (real) dimension $2n - 2$ which induces the 0-class in $H_{2n-2}(X;\mathbb{Z})$. We call Σ a *vanishing cycle*. For $\{\Gamma\} \in H_{2n-1}(X, \Sigma; \mathbb{Z})$ with

$$\partial\Gamma = \Sigma,$$

we consider Γ as a $(2n-1)$-current. We endow X with the Riemannian metric induced from the Fubini-Study metric on $\mathbb{P}\left(H^0(\mathcal{O}_X(1))^\vee\right)$. We have that $\Delta_d = \Delta_{d^c}$ and the orthogonal decomposition

$$\widehat{\mathcal{A}}_X^* = \Delta_d G_d \widehat{\mathcal{A}}_X^* \oplus \mathcal{H}^* = \Delta_{d^c} G_{d^c} \widehat{\mathcal{A}}_X^* \oplus \mathcal{H}^*$$

so that, as a (rectifiable) current,

$$
\begin{aligned}
\Sigma &= dd^* G_d \left(\Sigma \right) \\
&= dd^* G_d \left(d^C \left(d^C \right)^* G_d \left(\Sigma \right) + \left(d^C \right)^* d^C G_d \left(\Sigma \right) \right) \\
&= dd^C \left(dd^C \right)^* G_d^2 \left(\Sigma \right) - d \left(dd^C \right)^* G_d^2 \left(d^C \left(\Sigma \right) \right).
\end{aligned}
\tag{1}
$$

Notice that, if the current Σ is d^C-closed, e.g. if Σ is a complex-analytic $(n-1)$-cycle, then

$$
\Sigma = dd^C \left(dd^C \right)^* G_d^2 \left(\Sigma \right).
\tag{2}
$$

In general we define

$$
\omega_\Sigma = d^* G_d \left(\Sigma \right).
$$

Thus

$$
\omega_\Sigma \big|_{X - |\Sigma|}
$$

is closed with integral periods and has integral 1 against the boundary of a small oriented ball transverse to Σ and so, if Σ is connected, it can be thought of as the positively oriented generator of the cokernel of the map

$$
H^{2n-1} \left(X; \mathbb{Z} \right) \to H^{2n-1} \left(X - |\Sigma|; \mathbb{Z} \right).
$$

In the general case we want the element of the cokernel represented in $H^0 \left(|\Sigma|; \mathbb{Z} \right)$ by a cycle consisting of a single point on each component of Σ.

Next we have the harmonic decomposition

$$
\begin{aligned}
\Gamma &= d^* d G_d \left(\Gamma \right) + dd^* G_d \left(\Gamma \right) + \psi_\Sigma \\
&= d^* G_d \left(\Sigma \right) + dd^* G_d \left(\Gamma \right) + \psi_\Sigma \\
&= \omega_\Sigma + dd^* G_d \left(\Gamma \right) + \psi_\Sigma
\end{aligned}
\tag{3}
$$

where $\psi_\Sigma \in \mathcal{H}^{2n-1}$. For any $\eta \in \mathcal{H}^{2n-1}$ we have

$$
\int_X \psi_\Sigma \wedge \eta = \int_\Gamma \eta.
$$

If, in (3) Γ is replaced by another chain Γ' with harmonic projection ψ_Σ' and such that $\partial \Gamma' = \Sigma$, then $d \left(\Gamma - \Gamma' \right) = 0$ so that

$$
\Gamma - \Gamma' = dd^* \left(\Gamma - \Gamma' \right) + \psi_\Sigma - \psi_\Sigma'
$$

and so

$$
\left(\psi_\Sigma - \psi_\Sigma' \right) \in \mathcal{H}^{2n-1}
$$

is an integral class.

Definition 2 *The element $\left\{ h \mapsto \int_X \psi_\Sigma \wedge h \right\} \in J\left(X \right)$ is called the Abel-Jacobi image of Σ, which we denote as α_Σ. The element $\{ \psi_\Sigma \} \in J\left(X \right)^\vee$ will be denoted as β_Σ.*

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Notice that β_Σ the image of α_Σ under the isomorphism

$$J(X) \to J(X)^\vee$$

induced by the Poincaré duality and that

$$\beta_\Sigma : H_{2n-1}(X;\mathbb{Z}) \to \frac{\mathbb{R}}{\mathbb{Z}} \cong U(1)$$

exactly defines a flat line bundle L_{β_Σ} on $J(X)$ with (flat) $U(1)$-connection.

Next consider the expression

$$\int_{\Gamma'} \omega_\Sigma.$$

Fixing $\Sigma' = \partial\Gamma'$ the quantity $\int_{\Gamma'} \omega_\Sigma$ is only well-defined modulo

$$\left\{ \int_{\Theta'} \omega_\Sigma : \{\Theta'\} \in H_{2n-1}(X - |\Sigma|;\mathbb{Z}) \right\}.$$

Since $d(\omega_\Sigma) = \Sigma$, the local periods around $|\Sigma|$ are integers, hence $\left\{ \int_{\Gamma'} \omega_\Sigma \right\} \in \frac{\mathbb{R}}{\mathbb{Z}}$ is well-defined modulo

$$G_\Sigma := \left\{ \int_{\Theta'} \omega_\Sigma : \{\Theta'\} \in H_{2n-1}(X;\mathbb{Z}) \right\}.$$

Definition 3 *Given any two smoothly embedded oriented disjoint submanifolds $\Sigma, \Sigma' \subseteq X$ of (real) dimension $2n-2$ which each induce the 0-class in $H_{2n-2}(X;\mathbb{Z})$, we define a topological height pairing*

$$(\Sigma.\Sigma') = \int_{\Gamma'} \omega_\Sigma \in \frac{\mathbb{R}}{G_\Sigma + \mathbb{Z}}. \tag{4}$$

We extend both of the definitions just above to cycles

$$\Sigma = \sum_i m_i \Sigma_i$$

by linearity. In Definition 3 ω_Σ is a current determined by a smooth form on $X - |\Sigma|$. Thus we are allowed to integrate against a chain Γ' whose support is disjoint from $|\Sigma|$. In what follows, we will always assume such disjointness. Also if we integrate the wedge product of two currents, we will always restrict to the case in which the singular supports of the two currents are disjoint. Compare this definition with that of [Hai]. In the next two Sections we use the height pair to define a lifting of the point $(\alpha_\Sigma, \beta_{\Sigma'})$ to the fiber of the Poincaré bundle above that point.

If h denotes the harmonic projector, $\int_X \omega_\Sigma \wedge h(\Theta') = 0$ and

$$\Theta' - h(\Theta') = dd^* G_d \Theta'$$

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is exact on $X - |\Theta'|$. For the harmonic decompositions

$$\Gamma = dd^* G_d \left(\Gamma\right) + \omega_\Sigma + h\left(\Gamma\right)$$
$$\Theta' = dd^* G_d \Theta' + 0 + h\left(\Theta'\right)$$

we have

$$\Gamma \cdot \Theta' = -\int_{\Theta'} \omega_\Sigma + \int_\Gamma h\left(\Theta'\right) \tag{5}$$
$$\in \mathbb{Z}$$

as long as $|\Theta'|$ is disjoint from $|\Sigma|$. So $\int_{\Theta'} \omega_\Sigma$ is well-defined. Indeed this is exactly B. Harris's height pairing (Σ, Θ') in [Har].

To see what happens when Θ' is moved to cross Γ transversely, consider a one-parameter family $\Sigma'\left(t\right)$ of Σ' parametrized by a complex parameter t such that

$$\bigcup_{|t|<\varepsilon} \Sigma'\left(t\right) \tag{6}$$

forms a smooth submanifold of X fibered over the t-disk and meets Σ transversally at a point of $\Sigma'\left(0\right)$. Consider the real-valued function

$$f\left(t\right) = \left(\Sigma'\left(t\right).\Sigma\right) = -\int_{\Gamma'_t} \omega_\Sigma.$$

The manifold $\bigcup_{|t|=\varepsilon/2} \Sigma'\left(t\right)$ is homologous in $X - |\Sigma|$ to the boundary of a small ball in $\bigcup_{|t|=\varepsilon/2} \Sigma'\left(t\right)$ around the point $\Sigma'\left(0\right) \cap \Sigma$. Since $d\omega_\Sigma = \Sigma$ it follows immediately that

$$f\left(e^{2\pi i}t\right) - f\left(t\right) = \pm 1. \tag{7}$$

Thus the height pairing is well-defined as an element of $\frac{\mathbb{R}}{\mathbb{Z}}$.

Finally we compare $(\Sigma.\Sigma')$ and $(\Sigma'.\Sigma)$, at least modulo $\mathbb{Z}$ as follows. As in (5) we have

$$\Gamma \cdot \Gamma' = \int_\Gamma \omega_{\Sigma'} - \int_{\Gamma'} \omega_\Sigma + \int_\Gamma h\left(\Gamma'\right) \tag{8}$$
$$\in \mathbb{Z}$$

for $\partial \Gamma = \Sigma$ and $\partial \Gamma' = \Sigma'$ since $|\Sigma|$ and $|\Sigma'|$ are assumed disjoint.

4 Cohomology filtered by weight

Again fix disjoint Σ and Σ'. In what follows we will assume $|\Sigma|$ and $|\Sigma'|$ connected. The general case follows by linearity. Analogously to [Hai], we wish to describe $(\Sigma.\Sigma')$ as the obstruction to splitting some kind of natural structure

on real cohomology. Since any $(2n-1)$-form restricts to zero on Σ and Σ' each harmonic form representing a class in $H^{2n-1}(X;\mathbb{Z})$ has a canonical lifting to $H^{2n-1}(X,|\Sigma|;\mathbb{R})$ and to $H^{2n-1}(X,|\Sigma'|;\mathbb{R})$. Now $H^{2n-1}(X-|\Sigma|,|\Sigma'|;\mathbb{R})$ is the $(2n-1)$-cohomology of the deRham complex

$$\left(\mathcal{A}_{|\Sigma'|}^{*-1}\oplus\mathcal{A}_{X-|\Sigma|}^{*},D\right)$$

giving the mapping cone associated to the inclusion

$$|\Sigma'|\to X-|\Sigma|$$

Since any $(2n-1)$-form restricts to zero on Σ, the harmonic representatives $\mathcal{H}_{\mathbb{R}}^{2n-1}(X)$ for $H^{2n-1}(X;\mathbb{R})$ and $\mathcal{H}_{\mathbb{R}}^{2n-2}(|\Sigma'|)$ for $H^{2n-2}(|\Sigma'|;\mathbb{R})$ then allow us to define a distinguished isomorphism

$$H^{2n-1}(X-|\Sigma|,\Sigma';\mathbb{R})\cong\mathbb{R}\cdot\omega_{\Sigma}\oplus\mathcal{H}_{\mathbb{R}}^{2n-1}(X)\oplus\mathcal{H}_{\mathbb{R}}^{2n-2}(|\Sigma'|). \qquad (9)$$

Now
$$H_{2n-1}(X-|\Sigma|,|\Sigma'|;\mathbb{R})\cong H^{2n-1}(X-|\Sigma|,\Sigma';\mathbb{R})^{\vee}$$

So via (9) we obtain

$$H_{2n-1}(X-|\Sigma|,|\Sigma'|;\mathbb{R})\cong(\mathbb{R}\cdot\omega_{\Sigma})^{\vee}\oplus\mathcal{H}_{\mathbb{R}}^{2n-1}(X)^{\vee}\oplus\mathcal{H}_{\mathbb{R}}^{2n-2}(|\Sigma'|)^{\vee}$$

and, writing

$$M_{\mathbb{R}}:=H_{2n-1}(X-|\Sigma|,|\Sigma'|;\mathbb{Z})$$
$$M_{\mathbb{R}}^{\vee}:=\mathbb{R}\cdot\omega_{\Sigma}\oplus\mathcal{H}_{\mathbb{R}}^{2n-1}(X)\oplus\mathcal{H}_{\mathbb{R}}^{2n-2}(|\Sigma'|),$$

we have an induced injection

$$M_{\mathbb{Z}}\to M_{\mathbb{R}} \qquad (10)$$
$$(\partial\Delta_{\Sigma},\{\Gamma_k\},\Gamma')\mapsto\left(\int_{\partial\Delta_{\Sigma}},\left\{\int_{\Gamma_k}\right\},\int_{\Gamma'}\right)$$

where Γ' is supported away from $|\Sigma|\cup|\Sigma'|$ and $\{\Gamma_k\}$ is an integral basis of $H_{2n-1}(X;\mathbb{Z})$ also supported away from $|\Sigma|\cup|\Sigma'|$ and Δ_{Σ} is a small real oriented $2n$-ball meeting $|\Sigma|$ transversely at one point.

Finally we can filter the $\mathbb{Z}$-module M via

$$W_2=M \qquad (11)$$
$$W_1=\ker\left(H_{2n-1}(X-|\Sigma|,|\Sigma'|;\mathbb{Z})\xrightarrow{\partial}H_{2n-2}(|\Sigma'|;\mathbb{Z})\right)$$
$$W_0=\operatorname{image}\left(H_0(|\Sigma|;\mathbb{Z})\to H_{2n-1}(X-|\Sigma|;\mathbb{Z})\to H_{2n-1}(X-|\Sigma|,|\Sigma'|;\mathbb{Z})\right)$$

give a three-step weight filtration to $H_{2n-1}(X-|\Sigma|,|\Sigma'|;\mathbb{Z})$.

5 Weight filtrations of real vector spaces defined over the integers

The last part of the previous section, in particular the mapping (10), motivates the following real analogue to (a special case of) the weight filtration in the theory of mixed Hodge structures.

Definition 4 *Let $M_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of rank $r + 2$ with a fixed filtration*

$$M_{\mathbb{Z}} = W_2 \supsetneqq W_1 \supsetneqq W_0 \supsetneqq \{0\}$$

such that W_2/W_1 and W_0 are free of rank one and W_1/W_0 is free of rank r. An integral structure ζ on

$$M_{\mathbb{R}} = M_{\mathbb{Z}} \otimes \mathbb{R}$$

is a filtration preserving morphism of $\mathbb{Z}$-modules

$$M_{\mathbb{Z}} \to M_{\mathbb{R}}$$

such that the induced maps

$$\frac{W_i}{W_{i-1}} \to \frac{W_i \otimes \mathbb{R}}{W_{i-1} \otimes \mathbb{R}}$$

are the tautological inclusions obtained by tensoring $\frac{W_i}{W_{i-1}}$ with $\mathbb{R}$.

For example, the inclusion

$$H_{2n-1}\left(X - |\Sigma|, |\Sigma'|; \mathbb{Z}\right) \to H_{2n-1}\left(X - |\Sigma|, |\Sigma'|; \mathbb{R}\right) \tag{12}$$

with weight filtration (11) gives an integral structure via the isomorphism (9).

Definition 5 *Two integral structures ζ and ζ' on $M_{\mathbb{Z}} \otimes \mathbb{R}$ are equivalent if*

$$\zeta' = \zeta \circ \mu$$

for some filtration-preserving isomorphism

$$\mu : M_{\mathbb{Z}} \to M_{\mathbb{Z}}.$$

Notice that the restriction of an integral structure ζ to W_1 is exactly the same thing as a homomorphism

$$\beta_\zeta : \frac{W_1}{W_0} \to W_0 \otimes \mathbb{R}$$

and that equivalent integral structures ζ and ζ' induce the same homomorphism

$$\beta_\zeta = \beta_{\zeta'} : \frac{W_1}{W_0} \to \frac{W_0 \otimes \mathbb{R}}{W_0}.$$

In the same way the data of an integral structure ζ exactly induces on $M_{\mathbb{Z}}/W_0$ a homomorphism

$$\alpha_\zeta : \frac{W_2}{W_1} \to \frac{W_1 \otimes \mathbb{R}}{W_0 \otimes \mathbb{R}}$$

and equivalent integral structures ζ and ζ' induce the same homomorphism

$$\alpha_\zeta = \alpha_{\zeta'} : \frac{W_2}{W_1} \to \frac{W_1 \otimes \mathbb{R}}{(W_0 \otimes \mathbb{R}) + W_1}.$$

Theorem 6 *For the integral structure ζ given in (12),*

$$\alpha_\zeta = \alpha_{\Sigma'} \in J(X)$$
$$\beta_\zeta = \beta_\Sigma \in J(X)^\vee.$$

The set of equivalence classes of integral structures that induce the pair $(\alpha_{\Sigma'}, \beta_\Sigma)$ is naturally isomorphic to the fiber of the topological Poincaré bundle at $(\alpha_{\Sigma'}, \beta_\Sigma)$.

The integral structure (12) is determined by $(\alpha_{\Sigma'}, \beta_\Sigma)$ and the topological height pairing $(\Sigma.\Sigma')$. Thus $(\Sigma.\Sigma')$ gives a distinguished lifting of the point $(\alpha_{\Sigma'}, \beta_\Sigma)$ to the fiber of the (topological) Poincaré bundle at $(\alpha_{\Sigma'}, \beta_\Sigma)$.

Proof. Recall from Definition 1 that the Poincaré circle bundle is defined by the pasting rule

$$(\alpha, \beta, r) \mapsto (\alpha + m, \beta + n, r + n(\alpha) + \beta(m))$$

under the action of $(m, n) \in H_{2n-1}(X; \mathbb{Z}) \times H^{2n-1}(X; \mathbb{Z})$ on $H_{2n-1}(X; \mathbb{R}) \times H^{2n-1}(X; \mathbb{R})$. Let $\{h_j\}$ be a basis of $\mathcal{H}_{\mathbb{Z}}^{2n-1}$, the module of harmonic forms with integral periods. That

$$\beta_\zeta = \beta_\Sigma$$

derives from the identity (5)

$$\int_{\Theta'} \omega_\Sigma = \int_X \psi_\Sigma \wedge h(\Theta') - \Gamma \cdot \Theta'.$$

In terms of the basis

$$\omega_\Sigma, \{h_j\}, \varepsilon_{\Sigma'}$$

for $M_{\mathbb{R}}^\vee$ in (9) the integral structure (10) is given by the period matrix

$$\begin{pmatrix} \int_{\partial \Delta_\Sigma} \omega_\Sigma & (0, \ldots, 0) & 0 \\ \left\{ \int_{\Gamma_k} \omega_\Sigma \right\} & \left\{ \int_{\Gamma_k} h_j \right\} & \begin{matrix} 0 \\ \ldots \\ 0 \end{matrix} \\ \int_{\Gamma'} \omega_\Sigma & \int_{\Gamma'} h_j & \int_{\partial \Gamma'} \varepsilon_{\Sigma'} \end{pmatrix}$$

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where the 'diagonal' entries in the above matrix are integers, the $(3,2)$-entry is $\alpha_{\Sigma'}$, $(2,1)$-entry is β_Σ, and the $(3,1)$-entry is $(\Sigma.\Sigma')$. The diffeomorphism group

$$\mathrm{Diff}\left(X, |\Sigma|, |\Sigma'|\right)$$

has a subgroup

$$\mathrm{Diff}_0\left(X, |\Sigma|, |\Sigma'|\right) := \ker\left(\mathrm{Diff}\left(X, |\Sigma|, |\Sigma'|\right) \to \mathrm{Aut}\left(H^{2n-1}\left(X;\mathbb{Z}\right)\right)\right).$$

$\mathrm{Diff}_0\left(X, |\Sigma|, |\Sigma'|\right)$ acts on the above matrix in such a way that the monodromy action associated to path γ in $\mathrm{Diff}_0\left(X\right)$ beginning at the identity diffeomorphism and ending in $\mathrm{Diff}_0\left(X, |\Sigma|, |\Sigma'|\right)$ acts trivially on the 'diagonal' entries and as

$$\left(\alpha_\Sigma, \beta_{\Sigma'}, r\right) \mapsto \left(\alpha_\Sigma + m_\gamma, \beta_{\Sigma'} + n_\gamma, r + n_\gamma\left(\alpha_\Sigma\right) + \beta_{\Sigma'}\left(m_\gamma\right)\right).$$

But modulo $\mathbb{Z}$ the action of γ on the entry $(3,1)$ is exactly

$$\int_{\Gamma'} \omega_\Sigma \mapsto \int_{\Gamma'} \omega_\Sigma + n_\gamma\left(\alpha_\Sigma\right) + \beta_{\Sigma'}\left(m_\gamma\right).$$

Said otherwise, the action of γ takes the integral structure

$$\zeta : M_\mathbb{Z} \to M_\mathbb{R}$$

to the equivalent integral structure

$$M_\mathbb{Z} \xrightarrow{\gamma_*} M_\mathbb{Z} \xrightarrow{\zeta} M_\mathbb{R}.$$

∎

6 Topological Jacobi inversion

Finally, as in the introduction, consider two primitive classes $\eta, \eta' \in H^{2n}\left(W;\mathbb{Z}\right)$ on a complex manifold W. Let $Z, Z' \subseteq W$ be real $2n$-manifolds representing their Poincaré duals. Let $\mathbb{L} \subset \mathbb{P}$ be a generic line (Lefschetz pencil) with $\mathbb{L}^{sm} = \mathbb{L} \cap \mathbb{P}^{sm}$ and let

$$\tilde{Z}, \tilde{Z}'$$

be liftings of Z, Z' into $X_\mathbb{L}$ to manifolds that in general position with respect to the fibration

$$\chi : X_\mathbb{L} \to \mathbb{L}$$

and to each other. Thus $\tilde{Z}$ and $\tilde{Z}'$ intersect transversely over m distinct points $p_j \in \mathbb{L}^{sm}$. Define

$$\Sigma_p = X_p \cap \tilde{Z}$$
$$\Sigma'_p = X_p \cap \tilde{Z}'.$$

(Notice that Σ_p or Σ'_p may be slightly singular, but only at a finite number of points of $\mathbb{L}^{sm} - \cup_j \{x_j\}$. These singularities will not affect any of the above computations since for them Σ_p and Σ'_p need only be rectifiable currents at these points.) The computation leading to (7) then shows that the monodromy of the height pairing computes the local obstruction to separating Σ_{p_j} and Σ'_{p_j}, that is, the local intersection number of $\tilde{Z}$ and $\tilde{Z}'$.

Part II

Analytic setting

7   The case in which Σ is of type (n, n)

From now on suppose that Σ and Σ' are analytic $(n-1)$-cycles. Although the choice will not matter for the purposes of what follows, we will endow $J(X)$ with the (Weil) complex structure given by the operator

$$-C : H^{2n-1}(X; \mathbb{R})^{\vee} \to H^{2n-1}(X; \mathbb{R})^{\vee}.$$

In this case, the topological height pairing $(\Sigma.\Sigma')$ is the real part of a complex-valued function

$$\int_{\Gamma'} \left(d^* + i \left(d^C \right)^* \right) G(\Sigma) = \frac{1}{2} \int_{\Gamma'} \overline{\partial}^* G(\Sigma).$$

Referring to (2) the imaginary part of this function is the classical height pairing as defined by Beilinson, Bolch and Gillet-Soulé. We denote this last pairing as $[\Sigma.\Sigma']$. Finally we call

$$\langle \Sigma.\Sigma' \rangle := (\Sigma.\Sigma') + i[\Sigma.\Sigma']$$

the *holomorphic height pairing* for reasons explained by the next Lemma.

Suppose now that

$$\Sigma'(t) = q_* p^*(\{t\}) - q_* p^*(\{t_0\})$$

for a flat family

$$I_T \subseteq T \times X \xrightarrow{q} X$$
$$\downarrow p$$
$$T$$

of effective analytic $(n-1)$-cycles where T is a smooth, connected projective manifold and $t, t_0 \in T$. In what follows we will want to handle the case in which $\dim(|\Sigma| \cap |\Sigma'(t)|) > 0$ for some $t \in T$ and still be able to compute an incidence divisor for Σ on T, at least up to rational equivalence. We do this by using [FL] to deform Σ in its rational equivalence class to a cycle $\hat{\Sigma}$ such that

$$\left\{ t : \dim \left(\left| \hat{\Sigma} \right| \cap |\Sigma'(t)| \right) > 0 \right\}$$

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is of codimension at least 2 in T. Then the incidence divisor

$$p_* \left(q^* \left(\hat{\Sigma} \right) \cdot I_T \right)$$

is well-defined and its rational equivalence class is independent of the choice of rationally equivalent deformation $\hat{\Sigma}$.

To check that

$$\alpha \left(\Sigma \left(t \right) - \Sigma \left(t_0 \right) \right)$$

is holomorphic in t at a general element $t \in T$, we may suppose that $\Sigma \left(t \right)$ is smooth and irreducible there and that T is a disk of complex dimension one. So, for any $h' \in \mathcal{H}^{2n-1} = \mathcal{H}^{2n-1} \left(X; \mathbb{R} \right)$,

$$h'|_{I_T} = \overline{h^{n-1,n}} + h^{n-1,n}$$

has pure type $(n, n-1) + (n-1, n)$. Since I_T is locally the product of T with an open set on $\Sigma \left(t \right)$ the form h' can be written as

$$\omega \wedge dt + \omega \wedge \overline{dt} + \beta \wedge dt \wedge \overline{dt}$$

with ω real of type $(n-1, n-1)$. Then, by the Cartan-Lie formula,

$$\overline{\frac{\partial}{\partial t}} \int_{\Gamma(t)} \left(h' - iCh' \right) = \int_{\Sigma(t)} \left\langle \overline{\frac{\partial}{\partial t}} \middle| \left(h' - iCh' \right) \right\rangle$$
$$= \int_{\Sigma(t)} \omega - \omega = 0.$$

In particular we have the following.

So in particular, the image of $CH^n_{alg} \left(X \right)$ is a complex subtorus A_X of the Weil complex torus $J \left(X \right)$ (and of the Griffiths complex torus $J \left(X \right)$).

Lemma 7 *For any $h' \in \mathcal{H}^{2n-1}$, the image*

$$p_* \left(q^* \left(h' \right) |_{I_T} \right)$$

is the sum of a globally defined holomorphic one-form on T and its conjugate.

We will also need an extension of that standard fact about the holomorphicity of the Abel-Jacobi map.

Lemma 8 *The differential*

$$d \int_{\Sigma'(t_0)}^{\Sigma'(t)} \overline{\partial}^* G \left(\Sigma \right)$$

is holomorphic on $\left(T - |p_* q^* \left(\Sigma \right)| \right)$ with logarithmic poles and integral residues at components of $|p_* q^* \left(\Sigma \right)|$.

Proof. It will suffice to prove the Lemma in the case that T is an algebraic curve.

$$\int_{\Sigma'(t_0)}^{\Sigma'(t)} \left(d^* + i \left(d^C \right)^* \right) G(\Sigma)$$

$$= \frac{1}{2} \int_{\Sigma'(t_0)}^{\Sigma'(t)} \overline{\partial}^* G(\Sigma).$$

Now $\overline{\partial}^* G(\Sigma)$ is of type $(n, n-1)$ and is d- and $\overline{\partial}$-closed off $|\Sigma|$. So by the Cartan-Lie formula

$$d \int_{\Sigma'(t_0)}^{\Sigma'(t)} \overline{\partial}^* G(\Sigma) = \left(\int_{\Sigma'(t)} \left\langle \frac{\partial}{\partial t} \middle| \overline{\partial}^* G(\Sigma) \right\rangle \right) dt + \left(\int_{\Sigma'(t)} \left\langle \overline{\frac{\partial}{\partial t}} \middle| \overline{\partial}^* G(\Sigma) \right\rangle \right) \overline{dt}$$

$$= \left(\int_{\Sigma'(t)} \left\langle \frac{\partial}{\partial t} \middle| \overline{\partial}^* G(\Sigma) \right\rangle \right) dt$$

by type. So $\langle \Sigma . \Sigma'(t) \rangle$ is a (multivalued) holomorphic function with real periods.

Now by [Le] or [Wa] formula (1.1), $[\Sigma . (\Sigma'(t))]$ has logarithmic growth as t approaches a point $t_1 \in |p_* q^*(\Sigma)|$. Alternatively, following [GS], §1.3, there is a log resolution Z of X with center $|\Sigma|$ on which the Green current of log type η_Σ with $d^* G(\Sigma) = d^C \eta_\Sigma$ can be written locally in the form

$$\eta_\Sigma \sim \sum_i a_i \log \left(s_i \overline{s_i} \right) + \beta$$

where the a_i and β are smooth forms ([GS], (1.3.2.1)). By further blow-ups supported over $|\Sigma|$, we can assume that Z is so constructed that the proper transform $\hat{I}_T$ of $q(I_T)$ meets the exceptional simple normal-crossing divisor transversely along a central fiber $\widetilde{\Sigma'_{t_1}}$. Then the inequality

$$C \left| \log (t - t_1) \overline{(t - t_1)} \right| + C' \geq \sum_i \left| \log \left(s_i \overline{s_i} \right) \right| \Big|_{\hat{I}_T}$$

for some positive constants C and C' reduces the logarithmic growth assertion to a calculus exercise.

Thus $[\Sigma . (\Sigma'(t))]$ is a well-defined, harmonic function of t on $(T - |p_* q^*(\Sigma)|)$ and has logarithmic growth at $|p_* q^*(\Sigma)|$, and so by an elementary theorem in one complex variable (e.g. [A], p. 165, ex. 2) we can locally write

$$[\Sigma . (\Sigma'(t))] = \lambda \cdot \log |t - t_1| + u(t)$$

where u is harmonic on a neighborhood of t_1. Thus locally

$$\langle \Sigma . (\Sigma'(t)) \rangle = \lambda \cdot 2\pi i \cdot \log (t - t_1) + v(t) \tag{13}$$

where v is holomorphic in a neighborhood of t_1. ∎

For example, suppose $X = T$ is a smooth projective curve and $q_* p^*(\{t\}) = \{t\}$. Let ψ_t be the unique meromorphic 1-form on T with real periods and

logarithmic poles at t and t_0 and residue 1 at t and residue -1 at t_0 and let γ_t be a path on T from t_0 to t. Then Lemma 8 just above implies that

$$\frac{1}{2}\overline{\partial}^* G\left(\{t\} - \{t_0\}\right) = \psi_t.$$

Then

$$t' \mapsto e^{2\pi i \int_{t_0}^{t'} \psi_t}$$

is a meromorphic section of the flat line bundle whose $U\left(1\right)$ representation is given by the periods of ψ_t. The divisor given by this section is $\{t\} - \{t_0\}$.

We continue to suppose that $\Sigma'\left(t\right)$ is given by a flat family

$$\left(q_* p^*\left(\{t\}\right) - q_* p^*\left(\{t_0\}\right)\right)$$

as above. $\left[\Sigma.\Sigma'\left(t\right)\right]$ is well-defined on $\left(T - \left|p_* q^*\left(\Sigma\right)\right|\right)$ and has logarithmic growth at $\left|p_* q^*\left(\Sigma\right)\right|$. $\left(\Sigma.\left(\Sigma'\left(t\right) - \Sigma'\left(t_0\right)\right)\right)$ is locally well-defined on $\left(T - \left|p_* q^*\left(\Sigma\right)\right|\right)$ and has integral periods around $\left|p_* q^*\left(\Sigma\right)\right|$ since

$$d\left(d^* G\left(\Sigma\right)\right) = \Sigma$$

and Σ is an integral cycle.

Lemma 9 *Let $\Sigma \in CH^n_{hom}\left(X\right)$ and let*

$$D_\Sigma := p_*\left(q^*\left(\Sigma\right) \cdot I_T\right).$$

Then

$$D_\Sigma = \left(e^{2\pi i \left\langle \Sigma . \left(\Sigma'\left(t\right) - \Sigma'\left(t_0\right)\right)\right\rangle}\right). \tag{14}$$

Proof. Again it suffices to consider the case in which T is a smooth curve. If $\Delta \subseteq T$ is a small disk around a point of $\left|p_*\left(q^*\left(\Sigma\right) \cdot I_T\right)\right|$, then with respect to the family

$$I_\Delta \subseteq \Delta \times X \xrightarrow{\ q\ } X$$
$$\downarrow p$$
$$\Delta$$

we have

$$\deg\left(\Delta \cdot D_\Sigma\right) = \deg\left(\Delta \cdot_\Delta p_*\left(q^*\left(\Sigma\right) \cdot I_T\right)\right)$$
$$= \deg\left(p^*\left(\Delta\right) \cdot_{\Delta \times X} q^*\left(\Sigma\right)\right)$$
$$= \deg\left(q_* p^*\left(\Delta\right) \cdot_X \Sigma\right).$$

Now use that, if $q\left(I_\Delta\right)$ and Σ intersect properly at $x_0 \in X$ and B_{x_0} is a ball around x_0 in X, then

$$\int_{\partial B_{x_0} \cap q(I_\Delta)} d^* G\left(\Sigma\right)$$

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equals the intersection multiplicity of $q\left(I_\Delta\right)$ and Σ at x_0. ■

Furthermore, following Narasimhan and Seshadri, holomorphic line bundles of degree 0 on $J\left(X\right)$ correspond to unitary representations

$$\pi_1\left(J\left(X\right),0\right)\to U\left(1\right).$$

(See [NS] §12. Alternatively see [Mum], pp. 86-87.) Said otherwise any element

$$\beta\in J\left(X\right)^\vee=\operatorname{Hom}\left(H_{2n-1}\left(X;\mathbb{Z}\right),\frac{\mathbb{R}}{\mathbb{Z}}\right)$$

$$=\operatorname{Hom}\left(\pi_1\left(J\left(X\right)\right),U\left(1\right)\right)$$

corresponds to a line bundle L_β with flat unitary connection.

Corollary 10 *Let* $\left\{\Sigma\right\}\in CH^n_{hom}\left(X\right)$. *Let*

$$L_{\beta_\Sigma}$$

denote the flat line bundle associated to β_Σ. *For any family* $\left\{\Sigma'\left(t\right)\right\}_{t\in T}$ *of algebraic* $(n-1)$-*cycles homologous to zero with* T *smooth, let*

$$T\xrightarrow{\alpha_T} J\left(X\right)$$

$$t\mapsto\left(q_*p^*\left(\gamma_t\right)\cdot I_T\right)$$

for $\partial\gamma_t=\left\{t\right\}-\left\{t_0\right\}$ *denote the Abel-Jacobi mapping. Then* $e^{2\pi i\left\langle\Sigma.\left(\Sigma'\left(t\right)\right)\right\rangle}$ *is a meromorpic section of* $\alpha_T^*\left(L_{\beta_\Sigma}\right)$ *and*

$$\alpha_T^*\left(L_{\beta_\Sigma}\right)=\mathcal{O}_T\left(D_\Sigma\right). \tag{15}$$

Proof. As in (5) we have modulo $\mathbb{Z}$ that

$$\int_{\Theta'} d^*G\left(\Sigma\right)\equiv\int_\Gamma h\left(\Theta'\right)$$

for any $\Theta'\in H_{2n-1}\left(X;\mathbb{Z}\right)$. In fact, if Θ' runs through an integral basis of $H_{2n-1}\left(X;\mathbb{Z}\right)$, $\left\{\int_X h_\Gamma\wedge h_{\Theta'}\right\}$ exactly describes the image of Σ under the

$$CH^n_{hom}\left(X\right)\xrightarrow{\alpha} J\left(X\right)$$

while $\int_{\Theta'} d^*G\left(\Sigma\right)$ exactly determines the periods of $d^*G\left(\Sigma\right)$ and therefore the $U\left(1\right)$-representation

$$\pi_1\left(J\left(X\right),0\right)\to U\left(1\right)$$

defining L_{β_Σ}. Since

$$\int_{\Theta'} d^*G\left(\Sigma\right)\equiv\int_X h_\Gamma\wedge h_{\Theta'}$$

for $\Theta'\in H_{2n-1}\left(X;\mathbb{Z}\right)$, we have by Lemma 9 that the meromorphic section $\left(e^{2\pi i\left\langle\Sigma.\left(\Sigma'\left(t\right)-\Sigma'\left(t_0\right)\right)\right\rangle}\right)$ of $\alpha_T^*\left(L_{\alpha\left(\Sigma\right)}\right)$ has associated divisor D_Σ. ■

Notice that, if X is a curve, $T=X=I_T$, and $\Delta\in CH_0\left(X\right)_{hom}$, then $\Delta=\Sigma\left(\Delta\right)=D_\Delta$ and the content of the above corollary is Abel's theorem.

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